The Market Price of Risk and the Equity Premium: A Legacy of the Great Depression?[*]

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Revised: March 2005

Abstract

Friedman and Schwartz hypothesized that the Great Depression created exaggerated fears of economic instability. We quantify their idea by using a robustness calculation to shatter a representative consumer's initial confidence in the parameters of a two-state Markov chain that truly governs consumption growth. The assumption that the consumption data come from the true Markov chain and the consumer's use of Bayes' law cause that initial pessimism to wear off. But so long as it persists, the representative consumer's pessimism contributes a volatile multiplicative component to the stochastic discount factor that would be measured by a rational expectation econometrician. We study how this component affects asset prices. We find settings of our parameters that make pessimism wear off slowly enough to allow our model to generate substantial values for the market price of risk and the equity premium.

KEY WORDS: Robustness, learning, asset pricing.

1 Introduction

The risk premium on a security depends on how much risk is to be borne and how much compensation a risk-averse agent requires to bear it. From the Euler equation for excess returns and the Cauchy-Schwartz inequality, Hansen and Jagannathan (1991) deduce an upper bound on expected excess returns,

$$E(R_x) \leq \frac{\sigma(m)}{E(m)}\sigma(R_x). \qquad (1)$$

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Here R_x represents excess returns, m is a stochastic discount factor, and $E(\cdot)$ and $\sigma(\cdot)$ denote the mean and standard deviation, respectively, of a random variable. The term $\sigma(R_x)$ represents the amount of risk to be borne, and the ratio $\sigma(m)/E(m)$ is the market price of risk.

Hansen and Jagannathan characterize the equity-premium puzzle in terms of a conflict that emerges between two ways of measuring or calibrating the market price of risk.[1] The first way of calibrating it is to contemplate thought experiments involving transparent and well-understood gambles. Those thought experiments usually suggest that agents are willing to pay only a small amount for insurance against gambles, implying that they are mildly risk averse.[2] When stochastic discount factor models are calibrated to represent those levels of risk aversion, the implied price of risk is typically small.

The second way to calibrate the market price of risk is to use asset market data on prices and returns along with equation (1) to estimate a lower bound on the market price of risk. This can be done without imposing any model for preferences. Estimates reported by Hansen and Jagannathan and Cochrane and Hansen (1992) suggest a price of risk that is so high that it can be attained in conventional models only if agents are very risk averse. The conflict between the two methods is thus that people seem to be risk tolerant when confronting transparent and well-understood gambles, yet their behavior in securities markets suggests a high degree of risk aversion.

There are a variety of reactions to this conflict. Some economists, like Kandel and Stambaugh (1991), Cochrane (1997), Campbell and Cochrane (1999), and Tallarini (2000), reject the thought experiments and propose models involving high degrees of risk aversion. Others put more credence in the thought experiments and introduce distorted beliefs to explain how a high price of risk can emerge in securities markets inhabited by risk-tolerant agents. This paper contributes to the second line of research. We study a standard consumption-based asset pricing model with agents who are mildly risk averse and examine how a small dose of initial pessimism affects its quantitative implications.

Our approach follows Friedman and Schwartz (1963), who expressed the idea that the Great Depression of the 1930s created a mood of pessimism that affected markets for money and other assets:

> "The contraction after 1929 *shattered* beliefs in a 'new era,' in the likelihood of long-continued stability ... . The contraction instilled instead an *exaggerated* fear of continued economic instability, of the danger of stagnation, of the possibility of recurrent unemployment." (p. 673, emphasis added).

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[1]See also section 6.6 of Hansen and Sargent (2000).

[2]For instance, see the Pratt calculations in Cochrane (1997, p. 17) or Ljunqvist and Sargent (2000, pp. 258-260). Kocherlakota (1996, p. 52) summarizes by stating that "a vast majority of economists believe that values for [the coefficient of relative risk aversion] above ten (or, for that matter, above five) imply highly implausible behavior on the part of individuals."

"[T]he climate of opinion formed by the 1930s ... [was] further strength-
ened by much-publicized predictions of 'experts' that war's end would be
followed by a major economic collapse. ...[E]xpectations of great instabil-
ity enhanced the importance attached to accumulating money and other
liquid assets." (p. 560).

Friedman and Schwartz attribute some otherwise puzzling movements in the velocity
of money in the U.S. after World War II to the gradual working off of pessimistic
views about economic stability that had been inherited from the 1930s.

"The mildness and brevity of the 1953-54 recession must have strongly
reinforced the lesson of the 1948-49 recession and reduced still further
the fears of the great economic instability. The sharp rise of velocity of
money from 1954 to 1957 – much sharper than could be expected on cycli-
cal grounds alone – can be regarded as a direct reflection of the growth
of confidence in future economic stability. The brevity of the 1957-58 re-
cession presumably further reinforced confidence in stability, but, clearly,
each such episode in the same direction must have less and less effect, so
one might suppose that by 1960 expectations were approaching a plateau.
... If this explanation should prove valid, it would have implications for
assets other than money." (pp. 674-675.)

Our story also posits that the Depression shattered confidence in a 'normal' set of
beliefs, making them more pessimistic in terms of their consequences for a represen-
tative consumer's utility functional, then explores how asset markets were affected
as pessimism gradually evaporated. But instead of studying velocity, we explore how
pessimism and learning influence the market price of risk.[3] From the robust control
literature, we adopt a particular forward-looking way of a taking a normal probabil-
ity law and from it deducing a pessimistic probability law that we use to describe
how confidence in that normal probability law was shattered, to use Friedman and
Schwartz's term.

The idea that pessimism can help explain the behavior of asset prices has already
been used in quantitative studies. Some papers study the quantitative effects on
asset prices by exogenously distorting peoples beliefs away from those that a ratio-
nal expectations modeler would impose; e.g., see Rietz (1988), Cecchetti, Lam, and
Mark (2000), and Abel (2002). Other papers endogenously perturb agents' beliefs
away from those associated with a rational expectations models. Thus, Hansen, Sar-
gent, and Tallarini (1999), Cagetti, Hansen, Sargent, and Williams (2002), Hansen,

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[3]Prima facie evidence that the Depression was influential can be found in Siegel (1992), who
reports that the equity premium rose from around 2 percent for the years 1802 to 1925 to 5.9
percent for the period 1926 to 1990. Although the assets used to calculate average returns are not
entirely comparable across periods, the estimates nevertheless lend credence to the idea that the
Depression marked a watershed in securities markets.

Sargent, and Wang (2002), and Anderson, Hansen, and Sargent (2003) study representative agents who share but distrust the same model that a rational expectations modeler would impute to them. Their distrust of it inspires the agents to make robust evaluations of continuation values by twisting their beliefs pessimistically relative to that model. This decision-theoretic model of agents who want robustness to model misspecification is thus one in which pessimistic beliefs are endogenous, i.e., they are outcomes of the analysis.

All of these papers assume pessimism that is *perpetual*, in the sense that the authors do not allow the agents in the models the opportunity to learn their ways out of their pessimism by updating their models as more data are observed. In acknowledging this feature of their models, Anderson, Hansen, and Sargent (2003) and Hansen, Sargent, and Wang (2002) calibrate the degree of robustness that a representative consumer wants, and the consequent quantity of pessimism that emerges, by requiring that the consumer's worst-case model be difficult to distinguish statistically from his approximating model by using a Bayesian model-detection test based on a finite sample of reasonable length.

In contrast, this paper assumes only transitory pessimism by allowing the representative consumer to update his model via Bayes's Law.[4] We distort the representative agent's initial ideas about transition probabilities away from those that a rational expectations modeler would impose. We calibrate a 'small dose' of initial pessimism by using the robustness and detection error probability approaches of Anderson, Hansen, and Sargent (2003) and Hansen, Sargent, and Wang (2002). Then we give the representative consumer Bayes's Law, which via a Bayesian consistency theorem eventually erases their pessimism. We ask: How do asset prices behave in the mean time?

2   The model

Our model combines features of several models. Following Mehra and Prescott (1985), we study an endowment economy populated by an infinitely-lived, representative agent. Our consumer has time-separable, isoelastic preferences,

$$U = E_0^s \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\alpha} - 1}{1 - \alpha}, \tag{2}$$

where C_t represents consumption, β is the subjective discount factor, and α is the coefficient of relative risk aversion. We set $\alpha = 0.25$ and $\beta = 0.985$, so that the consumers are mildly risk averse and reasonably patient.

The consumption good is produced exogenously and is nonstorable, so current-period output is consumed immediately. Realizations for gross consumption growth

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[4]Kurz and Beltratti (1997) and Kurz, Jin, and Motolese (2004) also study models with transitory belief distortions that they restrict according to the notion of a rational-beliefs equilibrium.

follow a two-state Markov process with high and low-growth states, denoted g_h and g_l, respectively. The Markov chain is governed by a transition matrix F, where $F_{ij} = \text{Prob}[g_{t+1} = j | g_t = i]$. Shares in the productive unit are traded, and there is also a risk-free asset that promises a sure payoff of one unit of consumption in the next period. Asset markets are frictionless, and asset prices reflect the expected discounted values of next period's payoffs,

$$P_t^e = E_t^s[m_{t+1}(P_{t+1}^e + C_{t+1})], \tag{3}$$
$$P_t^f = E_t^s(m_{t+1}).$$

The variable $m_{t+1} = \beta(C_{t+1}/C_t)^{-\alpha}$ is the consumer's intertemporal marginal rate of substitution, P_t^e is the price of the productive unit, which we identify with equities, and P_t^f is the price of the risk-free asset. Notice that we follow the Mehra-Prescott convention of equating dividends with consumption.[5]

The agent's subjective conditional-expectations operator is denoted E_t^s. Under rational expectations, we would equate this with the conditional-expectations operator implied by the true transition probabilities, F. To distinguish the two, we adopt the notation E_t^a to represent the expectations operator under the actual probabilities. It is well-known, however, that a rational-expectations version of this model cannot explain asset returns unless α and β take on values that many economists regard as implausible.[6] Therefore, we borrow from Cecchetti, Lam, and Mark (2000) (CLM) the idea that distorted beliefs ($E_t^s \neq E_t^a$) may help to explain asset-price anomalies. In particular, they demonstrate that a number of puzzles can be resolved by positing pessimistic consumers who over-rate the probability of the low-growth state. The consumers in our model also have pessimistic beliefs, at least temporarily.

Our approach differs from that of CLM in one important respect. Their consumers have permanently distorted beliefs, never learning from experience that the low-growth state occurs less often than predicted. In contrast, we assume that the representative consumer uses Bayes's theorem to update estimates of transition probabilities as realizations accrue. Thus, we also incorporate the idea of Barsky and DeLong (1993) and Timmerman (1993 and 1996) that learning is important for understanding asset prices.

In our model, a Bayesian consistency theorem holds, so the representative consumer's beliefs eventually converge to rational expectations. That means the market price of risk eventually vanishes because it is negligible in the rational-expectations version of the model. The question we explore concerns how long this takes. Our story begins circa 1940 with consumers who are just emerging from the Great Depression. We endow them with prior beliefs that exaggerate the probability of another catastrophic depression. Then we explore how their beliefs evolve and whether their

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[5]An asset entitling its owner to a share of aggregate consumption is not really quite the same as a claim to a share of aggregate dividends, so the equity in our model is only a rough proxy for actual stocks. That is one reason why we focus more on the market price of risk.

[6]For an excellent survey of attempts to model asset markets in this way, see Kocherlakota (1996).

pessimism lasts long enough to explain the price of risk over a length of time comparable to our sample of post-Depression data.

2.1   Objective Probabilities

We start with a hidden Markov model for consumption growth estimated by CLM. They posit that log consumption growth evolves according to

$$\Delta \ln C_t = \mu(S_t) + \varepsilon_t, \tag{4}$$

where S_t is an indicator variable that records whether consumption growth is high or low, and ε_t is an identically and independently distributed normal random variable with mean 0 and variance σ_ε^2. Applying Hamilton's (1989) Markov switching estimator to annual per capita US consumption data covering the period 1890-1994, CLM find the following:

Table 1: Maximum Likelihood Estimates of the Consumption Process

	F_{hh}	F_{ll}	μ_h	μ_l	σ_ε
Estimate	0.978	0.515	2.251	-6.785	3.127
Standard Error	0.019	0.264	0.328	1.885	0.241

Note: Reproduced from Cecchetti, et. al. (2000)

As CLM note, the high-growth state is quite persistent, and the economy spends most of its time there. Contractions are severe, with a mean decline of 6.785 percent per annum. Furthermore, because the low-growth state is moderately persistent, a run of contractions can occur with nonnegligible probability, producing something like the Great Depression. For example, the probability that a contraction will last 4 years is 7.1 percent, and if that were to occur, the cumulative fall in consumption would amount to 25 percent. In this respect, the CLM model resembles the crash-state scenario of Rietz (1988). The chief advantage relative to Rietz's calibration is that the magnitude of the crash and its probability are fit to data.

Notice also how much uncertainty surrounds the estimated transition probabilities, especially F_{ll}, the probability that a contraction will continue. This parameter is estimated at 0.515 with a standard error of 0.264. Using an asymptotic normal approximation, a 90 percent confidence interval ranges from 0.079 to 0.951, which implies that contractions could plausibly have median durations ranging from 3 months to 13 years.[7] Thus, even with 100 years of data, substantial model uncertainty endures. The agents in our model cope with this uncertainty.

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[7]We should distrust the asymptotic normal approximation for a transition probability. The point is just that the transition probabilities are hard to pin down precisely.

We simplify the endowment process by suppressing the normal innovation ε_t, assuming instead that gross consumption growth follows a two-point process,

$$
\begin{aligned}
g_t &= 1 + \mu_h/100 \quad \text{if } S_t = 1, \\
&= 1 + \mu_l/100 \quad \text{if } S_t = 0.
\end{aligned}
\tag{5}
$$

We retain CLM's point estimates of μ_h and μ_l as well as the transition probabilities F_{hh} and F_{ll}. We assume that this model represents the true but unknown process for consumption growth.[8]

2.2   Subjective Beliefs

To represent subjective beliefs, we assume that the representative consumer knows the two values for consumption growth, g_h and g_l, but does not know the transition probabilities F. Instead, he learns about the transition probabilities by applying Bayes's theorem to the flow of realizations.

The representative agent adopts a beta-binomial probability model for learning about consumption growth. A binomial likelihood is a natural representation for a two-state process such as this, and a beta density is the conjugate prior for a binomial likelihood. We assume that the agent has independent beta priors over (F_{hh}, F_{ll}),

$$
p(F_{hh}, F_{ll}) = p(F_{hh})p(F_{ll}),
\tag{6}
$$

where

$$
\begin{aligned}
p(F_{hh}) &\propto F_{hh}^{n_0^{hh}-1}(1 - F_{hh})^{n_0^{hl}-1}, \\
p(F_{ll}) &\propto F_{ll}^{n_0^{ll}-1}(1 - F_{ll})^{n_0^{lh}-1}.
\end{aligned}
\tag{7}
$$

The variable n_t^{ij} is a counter that records the number of transitions from state i to j through date t, and the parameters n_0^{ij} represent prior beliefs about the frequency of transitions. The likelihood function for a batch of data, $g^t = \{g_s\}_{s=1}^t$, is proportional to the product of binomial densities,

$$
p(g^t|F_{hh}, F_{ll}) \propto F_{hh}^{(n_t^{hh}-n_0^{hh})}(1 - F_{hh})^{(n_t^{hl}-n_0^{hl})} F_{ll}^{(n_t^{ll}-n_0^{ll})}(1 - F_{ll})^{(n_t^{lh}-n_0^{lh})},
\tag{8}
$$

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[8]The purpose of this modification is to simplify the Bayesian learning problem. For a hidden Markov specification with unknown transition probabilities, Bayesian updating would involve recursive application of something like Hamilton's maximum likelihood estimator, and that would be a substantial computational burden in the simulations we conduct below. By suppressing ε_t, we cast the learning problem in terms of a simple beta-binomial model, which makes Bayesian updating trivial. Brandt, Zeng, and Zhang (2004) study a closely-related Bayesian learning model with hidden states and known transition probabilities. We assume unknown transition probabilities, and that is what complicates the filtering problem.

where $(n_t^{ij} - n_0^{ij})$ is the number of transitions from state i to j observed in the sample.[9] Multiplying the likelihood by the prior delivers the posterior kernel,

$$p(F_{hh}, F_{ll}|g^t) \propto F_{hh}^{n_t^{hh}-1}(1 - F_{hh})^{n_t^{hl}-1}F_{ll}^{n_t^{ll}-1}(1 - F_{ll})^{n_t^{lh}-1}, \qquad (9)$$
$$\propto p(F_{hh}|g^t)p(F_{ll}|g^t),$$

where

$$p(F_{hh}|g^t) = beta(n_t^{hh}, n_t^{hl}), \qquad (10)$$
$$p(F_{ll}|g^t) = beta(n_t^{ll}, n_t^{lh}).$$

With independent beta priors over F_{hh} and F_{ll} and a likelihood function that is a product of binomials, the posteriors are also independent and have the beta form. The counters are sufficient statistics.

This formulation makes the updating problem trivial. Agents enter each period with a prior of the form (9). We assume they observe the state, so to update their beliefs they just need to update the counters, incrementing by 1 the element n_{t+1}^{ij} that corresponds to the realizations of g_{t+1} and g_t. The updating rule can be expressed as

$$n_{t+1}^{ij} = n_t^{ij} + 1 \quad \text{if } g_{t+1} = j \text{ and } g_t = i, \qquad (11)$$
$$n_{t+1}^{ij} = n_t^{ij} \quad \text{otherwise.}$$

Substituting the updated counters into (10) delivers the new posterior, which then becomes the prior for the following period. The date-t estimate of the transition probabilities is formed from the counters,

$$F_t = \begin{bmatrix} \frac{n_t^{hh}}{n_t^{hh}+n_t^{hl}} & \frac{n_t^{hl}}{n_t^{hh}+n_t^{hl}} \\ \frac{n_t^{lh}}{n_t^{lh}+n_t^{ll}} & \frac{n_t^{ll}}{n_t^{lh}+n_t^{ll}} \end{bmatrix}. \qquad (12)$$

This model satisfies the conditions of a Bayesian consistency theorem.[10] Posterior estimates eventually converge to the true transition probabilities, and the representative consumer acquires rational expectations in the limit. The speed of convergence is central to our results.

Also notice the absence of a motive for experimentation to hasten convergence. Our consumers are learning about an exogenous process that their behavior cannot affect,[11] so they engage in passive learning, waiting for 'natural experiments' to reveal the truth. The speed of learning depends on the rate at which these experiments occur. Agents learn quickly about features of the Markov chain that occur often, more slowly about features that occur infrequently.

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[9]According to this notation, n_t^{ij} represents the sum of prior plus observed counters.

[10]See appendix B of Gelman, Carlin, Stern, and Rubin (1995).

[11]Even if consumption were a choice variable, atomistic consumers would not experiment because actions that are decentralized and unilateral have a negligible influence on aggregate outcomes.

For CLM's endowment process, that means agents learn quickly about F_{hh}, for the economy spends most of its time in the high-growth state and there are many transitions from g_h to g_h. Because this is a two-state model and rows of F must sum to one, it follows that agents also learn quickly about $F_{hl} = 1 - F_{hh}$, the transition probability from the high-growth state to the contraction state. Even so, uncertainty about expansion probabilities is important for our story. In theory, the key variable is not the estimate $F_{ij}(t)$ but the ratio $F_{ij}(t)/F_{ij}$.[12] Even though $F_{hh}(t)$ moves quickly into the neighborhood of F_{hh}, uncertainty about $F_{hl}(t)/F_{hl}$ endures, simply because F_{hl} is a small number. Seemingly small changes in $F_{hl}(t)$ remain influential for a long time because a high degree of precision is needed to stabilize this ratio.

Learning about contractions is even more difficult. Contractions are rare, yet one must occur in order to update estimates of F_{ll} or $F_{lh} = 1 - F_{ll}$. Indeed, because the ergodic probability of a contraction is 0.0434,[13] a long time must pass before a large sample of contraction observations accumulates. The persistence of uncertainty about the contraction state is also important in the simulations reported below, for that also retards learning.

2.3   How Asset Prices are Determined

After updating beliefs using (10) and (11), the representative consumer makes investment decisions and market prices are determined. At this stage, we assume that our consumer adopts an 'anticipated utility' approach to decision making, as in Kreps (1998). In an anticipated-utility model, a decision maker recurrently maximizes an expected utility function that depends on a stream of future outcomes, with respect to a probability model that is recurrently reestimated. Although an anticipated utility agent learns, he abstracts from parameter uncertainty when making decisions. That is, parameters are treated as random variables when learning but as constants when formulating decisions. This is a widely used convention in the economic literature on convergence of least-squares learning to rational expectations and in parts of the applied mathematics literature on adaptive control. In the context of our model, this involves treating estimated transition probabilities as if they were constant and known with certainty when making decisions. In particular, when making multi-step forecasts, the representative consumer neglects that future probability estimates will be updated. Instead, at each date t, they use the current estimate F_t to make projections far into the future. This behavioral assumption can be regarded as a form of bounded rationality or as an approximation to a more complex, fully Bayesian decision problem.[14]

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[12]How the ratio comes into play is explained below.

[13]A contraction is not an ordinary recession; it is more like a deep recession or a depression.

[14]The chief obstacle in calculating the solution to a fully Bayesian problem is the curse of dimensionality. When viewed as an approximation, the anticipated-utility approach can be regarded as a strategy for managing the size of the state space. In a related example, Cogley and Sargent (2004)

On this assumption, prices are determined in the same way as in a rational expectations model, after substituting the current estimate F_t for the true transition matrix F. At each date t, we solve for prices by following the algorithm in Mehra and Prescott. First, write the Euler equation for equities as

$$P_t^e(S_t = i, C_t) = \beta \sum_{j=1}^{2} F_{ij}(t) g_{jt+1}^{-\alpha} [P_t^e(S_t = j, g_{jt+1} C_t) + g_{jt+1} C_t]. \tag{13}$$

Then use the fact that the equity price is homogenous of degree 1 in consumption, $P_t^e(S_t = i) = w_t(S_t = i) C_t$, to re-write this condition as

$$w_t(S_t = i) = \beta \sum_{j=1}^{2} F_{ij}(t) g_{jt+1}^{1-\alpha} [1 + w_t(S_t = j)]. \tag{14}$$

This is a system of n linear equations in n unknowns that can be solved for weights $w_t(S_t = i)$. With the weights in hand, one can calculate net equity returns as

$$r_{ij}^e(t) = \frac{g_{jt}[1 + w_t(S_t = j)]}{w_{t-1}(S_{t-1} = i)} - 1. \tag{15}$$

Similarly, the price of a risk-free bond is

$$P_t^f(S_t = i) = \beta \sum_{j=1}^{2} F_{ij}(t) g_{jt+1}^{-\alpha}, \tag{16}$$

and the risk-free rate is $r_{ft}(S_t = i) = 1/P_t^f(S_t = i) - 1$.

2.4 Shattering Beliefs: Calibrating the Representative Consumer's Pessimistic Prior

All that remains is to describe how we specify the representative consumer's prior. We inject an initial dose of pessimism by using a procedure from the robust control literature to deduce a worst-case transition model from CLM's estimated model. We assume that the representative consumer has a benchmark approximating model that coincides with the true transition probabilities. But we also suppose that the Depression shattered his confidence in that model in a particular way. Although we assume that the benchmark data actually governs the data, just as in a rational expectations model, we endow the representative consumer with a prior that is pessimistically distorted relative to the benchmark data. That puts pessimism into the representative consumer's evaluations of risky assets. As data accrue, the consumer's application of Bayes law causes his pessimism to dissolve.

We define a model that is distorted relative to the rational expectations benchmark F_{ij} as

$$F_{ij}^\tau = \tau_{ij} F_{ij} \tag{17}$$

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evaluate the quality of this approximation and find that it is excellent.

where τ_{ij} is a strictly positive random variable that satisfies $\sum_{ij} \tau_{ij} F_{ij} = 1$ for all i. According to (17), τ_{ij} serves as a Radon-Nikodým derivative for distorting the distribution over next period's state, conditional on being in growth state i now. Define the conditional entropy of the distortion as the expected log likelihood ratio,

$$I_i(\tau) = \sum_j \log \frac{F_{ij}^\tau}{F_{ij}} F_{ij}^\tau, \tag{18}$$

$$= \sum_j \log \tau_{ij} F_{ij}^\tau,$$

$$= \sum_j (\log \tau_{ij}) \tau_{ij} F_{ij}.$$

Notice the change of measure that occurs when moving from the second to the third line. To induce robust evaluations of continuation values, let $W(C, g_i)$ be a value function and consider the problem

$$W(C, g_i) = U(C) + \beta \inf_\tau \left[\sum_j W(g_j C, g_j) \tau_{ij} F_{ij} + \theta I_i(\tau) \right], \tag{19}$$

where $u(C) = C^{1-\alpha}/(1-\alpha)$ and $\theta > 0$ is a parameter that penalizes the minimizer for distortions with large conditional entropy. Later we pin down the parameter θ by calculating detection-error probabilities. The minimizer of this problem is

$$\tau_{ij}(C) \propto \exp\left(-\frac{W(Cg_j, g_j)}{\theta}\right). \tag{20}$$

When we use Whittle's (1990) risk-sensitivity parameter γ by setting $\gamma = -2\theta^{-1}$, the minimized value of (19) is the indirect value function[15]

$$W(C, g_i) = U(C) + \beta \frac{2}{\gamma} \log \sum_j \exp\left(\frac{\gamma}{2} W(g_j C, g_j)\right) F_{ij}. \tag{21}$$

We approximate $W(C, g_i)$ by a pair of 4th order polynomials, use least squares approximation, and iterate to convergence on (21). We then compute the 'twisting factor'

$$\tau_j^*(C) \propto \exp\left(\frac{\gamma}{2} W(Cg_j.g_j)\right), \tag{22}$$

We normalize C to be 1, and think of this choice as scaling consumption in 1940, the beginning of our computational experiment.[16] We then use the resulting τ to

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[15] We follow Hansen and Sargent (1995) rather than Whittle in the way we introduce discounting.

[16] Notice that the distortion depends on the level of C in a way that makes the distortion diminish with increases in C. His dissatisfaction with that feature of specifications like ours was the starting point for Pascal Maenhout's (2004) suggestion about specifying θ in a way that would eliminate that dependence.

compute worst-case transition probabilities,

$$F_{ij}^{WC} = \frac{\tau_j^* F_{ij}}{\sum_k \tau_k^* F_{ik}}. \tag{23}$$

We use this distortion to center the initial prior of our representative consumer.

To complete our specification of the representative consumer's prior, our last step is to translate the worst-case frequencies F_{ij}^{WC} into a prior number of counters n_0^{ij}. We suppose that the prior is based on a training sample of size T_0 and initialize the counters at $n_0^{ij} = (T_0/2)F_{ij}^{WC}$. This replicates the worst-case transition frequencies for a sample of T_0 observations.[17]

The prior depends on two free parameters γ and T_0 that govern the desired degree of robustness and tightness of initial beliefs, respectively. To discipline the degree of pessimism, we restrain γ so that the worst-case model is statistically hard to distinguish from the reference model in a sample of size T_0. Following Anderson, Hansen, and Sargent (2003) and Hansen, Sargent, and Wang (2002), we do this by applying a Bayesian model detection test.

This test is based on the log-likelihood ratio of the worst-case model relative to the benchmark model. According to equation (8), the log-likelihood ratio for a sample of size T_0 is

$$\log LR = \sum_i \sum_j n_{T_0}^{ij} \log(F_{ij}^{WC}/F_{ij}). \tag{24}$$

In a given sample, the benchmark model is more likely if $\log LR < 0$, and the worst-case model is more likely if $\log LR > 0$. A type I classification error occurs if the log-likelihood ratio happens to be positive when data are generated by the benchmark model, and a type II classification error occurs when the log-likelihood ratio is negative and the data are generated from the worst-case model. Assuming a prior probability of 1/2 for each model, the probability of a detection error is

$$0.5 \cdot [\text{Prob}(\log LR > 0|\text{Benchmark Model}) + \text{Prob}(\log LR < 0|\text{Worst-Case Model})]. \tag{25}$$

Through γ, the detection error probability depends on how much the reference and worst-case models disagree. Recall that $\gamma = 0$ reproduces an expected-utility model. Because there is no concern for robustness in that case, the two models coincide and the term in brackets equals 1. Thus, for $\gamma = 0$, the detection error probability is 0.5. As $|\gamma|$ increases, the worst-case model differs more and more from the reference model, and it becomes easier to classify data as coming from one or the other. Therefore the detection error probability falls as $|\gamma|$ increases. For a given T_0, we calibrate γ so that the detection error probability is still fairly substantial. In

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[17]This involves a slight abuse of concepts. The counters are supposed to be integers, but here they are real valued. We could round to the nearest integer, but when initial beliefs are diffuse (T_0 is small) this results in a substantial additional distortion of the prior. We prefer to preserve the worst-case transition probabilities at the cost of violating the integer constraint.

that way, we rule out initial scenarios in which the representative consumer guards against specification errors that could be easily dismissed based on observations in their training sample.

The next table summarizes the results of Monte Carlo simulations involving the CLM reference model and a variety of worst-case alternatives for various combinations of γ and T_0. Our model is annual, so T_0 refers to the number of years in a hypothetical training sample. In each case, we deduced the worst-case alternative for the specified value of γ by following the steps outlined above. Then we simulated 20,000 samples from the reference and worst-case models, evaluated log-likelihood ratios, and counted the proportion of type I and II errors.

Table 2: Detection Error Probabilities

	$T_0 = 10$	30	50	70
$\gamma = -0.08$	0.439	0.379	0.346	0.319
-0.09	0.419	0.349	0.308	0.278
-0.10	0.401	0.313	0.265	0.226
-0.11	0.376	0.270	0.211	0.172
-0.12	0.339	0.209	0.147	0.109
-0.13	0.277	0.143	0.084	0.050
-0.14	0.205	0.072	0.029	0.012
-0.15	0.126	0.025	0.006	0.001
-0.16	0.080	0.008	0.001	0.000
-0.17	0.038	0.002	0.000	0.000

Note: Entries for each (γ, T_0) combination are calculated by Monte Carlo simulations involving 20,000 draws from the reference and worst-case models.

Distinguishing the worst-case model from the reference model is difficult when the prior is diffuse (i.e., when T_0 is small) but becomes easier as the prior becomes more informative. For a given value of γ, the detection error probability falls as T_0 increases. Similarly, for a training sample of a given size, distinguishing the models is harder when $|\gamma|$ is small and becomes easier as $|\gamma|$ increases. Thus, the detection error probability also declines as we move down each column.

For the simulations reported below, we adopt a detection error probability of 10 percent and explore how the results vary with the tightness of the prior, which is indexed by T_0. By interpolating entries in table 2, this corresponds to $\gamma = -0.1556$ for $T_0 = 10$, $\gamma = -0.1360$ for $T_0 = 10$, $\gamma = -0.1275$ for $T_0 = 50$, and $\gamma = -0.1216$ for $T_0 = 70$.

Table 3 records the worst-case transition probabilities for these (γ, T_0) combinations. Relative to the true transition probabilities, which are reproduced in the last row, the representative consumer is initially pessimist both about the length

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of expansions and the length of contractions. That is, he underestimates F_{hh}, the conditional probability that an expansion will continue given that the economy is currently expanding, and he overestimates F_{ll}, the probability that a contraction will continue once one has already begun. It follows that the representative consumer also underestimates the ergodic probability of expansions and overestimates that of contractions. In other words, the consumer initially believes that contractions occur too often and are too long when they do occur. Since long contractions have the character of Great Depressions, our consumer is initially too wary of another crash.

Table 3: Worst-Case Transition Probabilities

	F_{hh}^{WC}	F_{ll}^{WC}
$T_0 = 10, \gamma = -0.1556$	0.775	0.932
$T_0 = 30, \gamma = -0.1360$	0.886	0.858
$T_0 = 50, \gamma = -0.1275$	0.914	0.817
$T_0 = 70, \gamma = -0.1216$	0.926	0.790
$\gamma = 0$	0.978	0.515

The worst-case priors resemble – at least qualitatively – one of the distorted-beliefs scenarios of CLM. They proposed two promising configurations for resolving asset pricing puzzles. One involved pessimism about expansions and contractions, along with a slight degree of risk aversion $\alpha < 1$, and β not too far below 1. The other scenario involved pessimism about expansions but optimism about contractions (i.e., F_{hh} and F_{ll} were both underestimated), along with a higher degree of risk aversion $\alpha \doteq 9$, and values of β around 0.84. Our robustness calculations point toward the first scenario but not the second. It is hard to motivate optimism about contractions by appealing to robustness.[18] We also found that the second configuration did not survive the introduction of learning. Thus, our model is closer in spirit to their first scenario.

3   Simulation Results

We simulate asset returns by drawing 1000 paths for consumption growth from the true Markov chain governed by F. Each trajectory is 70 years long, to imitate the approximate amount of time that has passed since the Great Depression.[19] We endow the consumer with a worst-case prior, then let him apply Bayes' law to each consumption-growth sequence. At each date t, he updates beliefs in the way described above, then makes multi-step forecasts using current estimates of transition probabilities. Prices that induce the consumer to hold the two securities follow from the subjective Euler equations.

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[18]There may, of course, be other motivations for contraction-state optimism.

[19]Think of this as mimicking the period 1935-2005.

3.1 Prices of Risk in the Learning Economy

Hansen and Jagannathan calculate a market price of risk in two ways. The first, which we label the 'required' price of risk, is inferred from security market data without reference to a model discount factor. According to equation (1), the price of risk must be as least as large as the Sharpe ratio for excess stock returns,

$$\frac{\sigma(m_t)}{E(m_t)} \geq \frac{E(R_{xt})}{\sigma(R_{xt})}. \tag{26}$$

Thus, the Sharpe ratio represents a lower bound that a model discount factor must satisfy in order to reconcile asset returns with an ex post, rational expectations Euler equation. Hansen and Jagannathan find that the required price of risk is quite large, on the order of 0.23.[20] Table 4 reproduces estimates in that ballpark using Shiller's annual data series for stock and bond returns.

Table 4: The Mean, Standard Deviation, and Sharpe Ratio for Excess Returns

	1872-2002	1872-1928	1929-2002	1929-1965	1966-2002
$E(R_{xt})$	0.0410	0.0266	0.0521	0.0708	0.0334
$\sigma(R_{xt})$	0.1734	0.1507	0.1892	0.2239	0.1474
$E(R_{xt})/\sigma(R_{xt})$	0.2364	0.1765	0.2754	0.3162	0.2266

Shiller's sample runs from 1872 to 2002, and for that period excess stock returns averaged 4.1 percent per annum with a standard deviation of 17.3 percent, implying a Sharpe ratio of 0.236. Before the Depression, however, the unconditional equity premium and Sharpe ratio were both lower. For the period 1872-1928, the mean excess return was 2.7 percent, the standard deviation was 15.1 percent, and the Sharpe ratio was 0.177. In contrast, after 1929 the equity premium and Sharpe ratio were 5.2 percent and 0.275, respectively. Furthermore, if the post-Depression period is split into two halves, we find that the equity premium and Sharpe ratio were higher in the first half, at 7.1 percent and 0.316, and somewhat lower in the second, at 3.3 percent and 0.223. Nevertheless, estimates of the bound hover around 0.25, which we take as our target to explain.

Hansen and Jagannathan also compute a second price of risk from discount factor models in order to check whether the lower bound is satisfied. They do this by substituting consumption data into a calibrated model discount factor and then computing its mean and standard deviation. For model prices of risk to approach the required price of risk, the degree of risk aversion usually has to be set very high. When it is set at more 'plausible' values, the model price of risk is quite small, often closer to 0.02 than to 0.2. Thus, the degree of risk aversion needed to explain security market

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[20]See also Cochrane and Hansen (1992) and Gallant, Hansen, and Tauchen (1991), who elaborate and extend their calculations.

data is higher than values that seem reasonable a priori. That conflict is evident in the rational expectations version of our model. Our stochastic discount factor is $m_{t+1} = \beta g_{t+1}^{-\alpha}$, and because our representative consumer is risk tolerant ($\alpha = 0.25$) the model price of risk under rational expectations is only 0.0048, too small by a factor of 50.

In a rational expectations model, there is a unique model price of risk because subjective beliefs coincide with the actual law of motion. But that is not the case in a learning economy. In our model, subjective beliefs eventually converge to the actual law of motion, but they differ along the transition path, so when we speak of a model price of risk we must specify the probability measure with respect to which moments are evaluated. At least two prices of risk are relevant in a learning economy, depending on the probability measure that is used to evaluate the mean and standard deviation of m_t.

If we asked the representative consumer about the price of risk, his response would reflect his beliefs. We call this the 'subjective' price of risk,

$$PR_t^s = \frac{\sigma_t^s(m_{t+1})}{E_t^s(m_{t+1})}. \tag{27}$$

Here a superscript s indicates that moments are evaluated using subjective probabilities. We focus initially on an unconditional measure of PR_t^s because that is what the unconditional Sharpe ratios in table 4 bound. By 'unconditional,' we mean that the mean and standard deviation of m_{t+1} do not depend on the state at date t. Time subscripts are still required, however, because subjective transition probabilities are updated from period to period. Changing beliefs cause unconditional moments to vary over time, making a learning economy non-stationary.

To calculate PR_t^s, we must evaluate the date-t unconditional mean and standard deviation in (27). Conditional on the state at t, the first and second moments are

$$E_t^s(m_{t+1}|s_t) = \sum\nolimits_{i=1}^{2} F_{ij}(t)m_i(t+1), \tag{28}$$

$$E_t^s(m_{t+1}^2|s_t) = \sum\nolimits_{i=1}^{2} F_{ij}(t)m_i^2(t+1).$$

If we invoke the anticipated-utility assumption that F_t is constant, we can approximate unconditional moments by weighted averages of conditional moments. With that assumption, we compute the vector of unconditional probabilities F_t^U associated with the current transition matrix F_t and then calculate unconditional first and second moments as

$$E_t^s(m_{t+1}) = \sum\nolimits_{i=1}^{2} F_i^U(t) \left[\sum\nolimits_{j=1}^{2} F_{ij}(t)m_j(t+1) \right], \tag{29}$$

$$E_t^s(m_{t+1}^2) = \sum\nolimits_{i=1}^{2} F_i^U(t) \left[\sum\nolimits_{j=1}^{2} F_{ij}(t)m_j^2(t+1) \right].$$

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To evaluate PR_t^s, we substitute (29) into

$$PR_t^s = \frac{[E_t^s(m_{t+1}^2) - E_t^s(m_{t+1})^2]^{1/2}}{E_t^s(m_{t+1})}. \tag{30}$$

Next, we imitate Hansen and Jagannathan by seeking the market price of risk needed to reconcile equilibrium returns with a rational-expectations Euler equation. In the learning economy, returns satisfy the subjective Euler equations (3), which we re-write as

$$E_t^s(m_{t+1}R_{t+1}|s_t) = \sum\nolimits_{j=1}^2 F_{ij}(t)m_j(t+1)R_{ij}(t+1) = 1. \tag{31}$$

But they do not satisfy the objective Euler equation $E_t^a(m_{t+1}R_{t+1}|s_t) = 1$ because the subjective and objective expectations operators disagree. To reconcile equilibrium returns with objective probabilities, we must apply a change of measure in (31),

$$1 = \sum\nolimits_{j=1}^2 F_{ij}\left(\frac{F_{ij}(t)}{F_{ij}}\right)m_j(t+1)R_{ij}(t+1), \tag{32}$$
$$= E_t^a(m_{t+1}^* R_{t+1}|s_t).$$

Notice how the change of measure twists the stochastic discount factor, transforming $m_j(t+1)$ into

$$m_{ij}^*(t+1) = m_j(t+1) \cdot (F_{ij}(t)/F_{ij}). \tag{33}$$

The extra term is the Radon-Nikodým derivative of the subjective transition probabilities with respect to the actual transition probabilities. Equation (32) is a rational expectations Euler equation that explains returns from the learning economy. Therefore, the price of risk that reconciles returns with rational expectations is

$$PR_t^{RE} = \frac{\sigma_t^a(m_{t+1}^*)}{E_t^a(m_{t+1}^*)}. \tag{34}$$

We calculate the RE price of risk by following the steps leading up to equation (27), but now substituting the twisted discount factor m_{t+1}^* for the consumers' IMRS and the actual transition probabilities F_{ij} for the estimated transition matrix. Conditional on the state at t, the first and second moments of m_{t+1}^* are

$$E_t^a(m_{t+1}^*|s_t) = \sum\nolimits_{i=1}^2 F_{ij}m_i^*(t+1) = \sum\nolimits_{i=1}^2 F_{ij}\frac{F_{ij}(t)}{F_{ij}}m_i(t+1), \tag{35}$$

$$E_t^a(m_{t+1}^{*2}|s_t) = \sum\nolimits_{i=1}^2 F_{ij}m_i^{*2}(t+1) = \sum\nolimits_{i=1}^2 F_{ij}\left(\frac{F_{ij}(t)}{F_{ij}}\right)^2 m_i^2(t+1).$$

Next, take unconditional averages of the conditional moments, using the ergodic probabilities F^U associated with the actual transition probabilities F:

$$E_t^a(m_{t+1}^*) = \sum_{i=1}^{2} F_i^U \left[\sum_{i=1}^{2} F_{ij} \frac{F_{ij}(t)}{F_{ij}} m_i(t+1) \right], \tag{36}$$

$$E_t^a(m_{t+1}^{*2}) = \sum_{i=1}^{2} F_i^U \left[\sum_{i=1}^{2} F_{ij} \left(\frac{F_{ij}(t)}{F_{ij}} \right)^2 m_i^2(t+1) \right].$$

Then substitute (36) into

$$PR_t^{RE} = \frac{[E_t^a(m_{t+1}^{*2}) - E_t^a(m_{t+1}^*)^2]^{1/2}}{E_t^a(m_{t+1}^*)}, \tag{37}$$

to calculate PR_t^{RE}.

In a learning economy, there is no reason why the two prices of risk, (27) and (34), must agree. They refer to different discount factors and are evaluated with respect to different transition probabilities. The existence of two model prices of risk and the fact that they disagree are the key to our resolution of the price-of-risk paradox. In our simulations, subjective prices of risk are quite small, in accordance with thought experiments and surveys, but RE prices of risk are large, reflecting the change of measure needed to reconcile returns from a learning economy with rational expectations.

Figure 1 portrays simulations of the two prices of risk from our learning economy. The four panels refer to simulations initialized with different priors. The upper-left panel, labeled $T_0 = 10$, refers to a vague and pessimistic prior based on an initial sample of size 10. The other panels progressively strengthen the prior – and shrink the degree of initial pessimism – with initial samples of 30, 50, and 70, respectively. In each panel, the solid line near zero depicts the subjective price of risk, PR_t^s, and the dashed curve illustrates the rational-expectations price of risk, PR_t^{RE}. Each line represents the cross-sectional average of the price of risk in a given year.[21]

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[21]The simulation consists of 5000 sample paths of length 70, and there are two prices of risk on each path at each date. The figure illustrates the date-t average across sample paths.
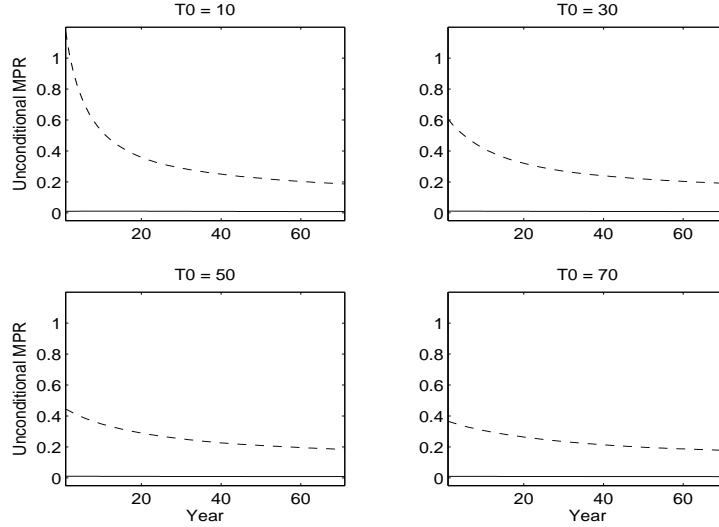
Figure 1: Subjective and RE Prices of Risk. Dashed lines portray PR_t^{RE} and solid lines PR_t^s.

The subjective price of risk is indeed very small, ranging from 0.01 at the beginning of the simulation to 0.008 at the end. The small values reflect the risk tolerance of our consumers, whose coefficient of relative risk aversion is just 0.25. These numbers are comparable to model prices of risk that Hansen, Jagannathan, and others calculate.

In contrast, PR_t^a is quite high. The rational-expectations price of risk starts out at values ranging from 0.4 to 1.0 – depending on the consumer's priors – and then declines gradually as time passes.[22] The decline reflects the decreasing importance of the Radon-Nikodým derivatives $(F_{ij}(t)/F_{ij})$, which eventually converge to 1 as subjective beliefs converge to objective probabilities. But convergence is slow: after 70 years, the RE price of risk is still quite a bit larger than the subjective price of risk, with mean estimates clustering around 0.185.

That is about 25 percent short of the benchmark value of 0.25. Nevertheless, although the mean estimate falls a bit short, a substantial fraction of sample paths have prices of risk that exceed the bound. Figure 2 portrays that fraction for various years. Virtually all the sample paths exceed the bound at the beginning of the simulation, the fraction falls to around 0.4 - 0.6 in the middle, and then settles near 0.2 at the end. Thus, RE prices of risk of 0.25 or more are not unusual in our model, even at the end of simulation.

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[22]Remember that the benchmark value of 0.25 is a lower bound, not a point estimate. Values greater than the bound do not necessarily refute the model.
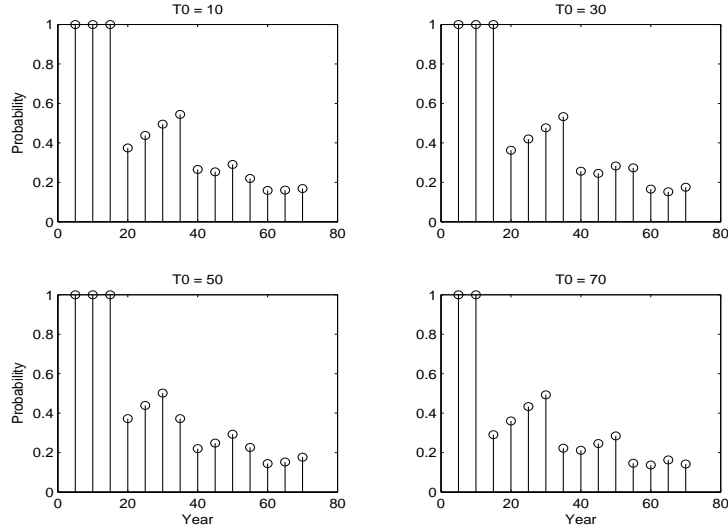
Figure 2: Probability that $PR_t^{RE} > 0.25$

Figure 3 shows how model prices of risk vary across expansions and contractions. Conditional prices of risk are calculated in the same way as above, except using conditional means and standard deviations from equations (28) and (35) instead of unconditional moments. Dashed lines still portray RE prices of risk, and solid lines illustrate subjective prices of risk. Circles mark contractions, and plus signs represent expansions. The figure again records the cross-sectional average of prices of risk at each date.
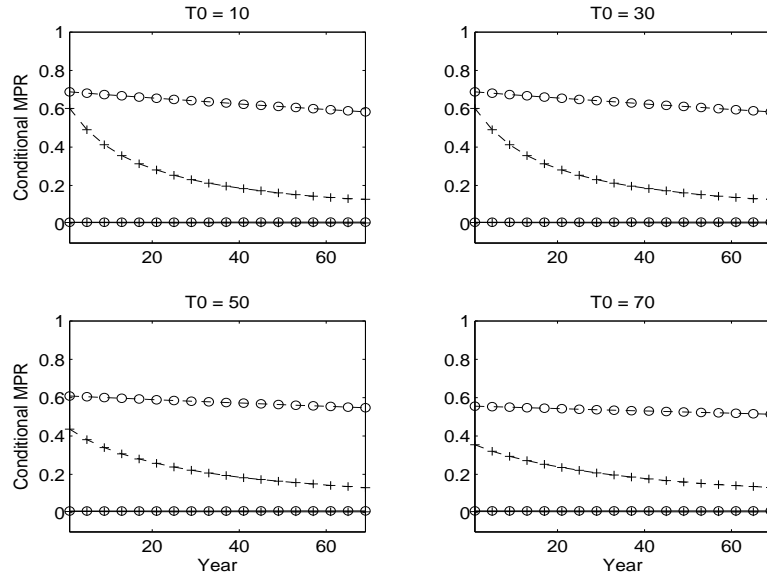
Figure 3: Conditional Prices of Risk. Dashed lines portray PR_t^{RE} and solid lines PR_t^s. Circles mark contractions, and plus signs represent expansions.

Subjective prices of risk are low in both expansions and contractions; indeed, the values differ so little across states that the two solid line lie on top of one another. The RE price of risk, on the other hand, varies more across states and is substantially higher in contractions. Notice also that the contraction-state price of risk falls more slowly than that for expansions. The persistence of the contraction value follows from the fact that the representative consumer learns more slowly about the contraction-state transition probabilities F_{lj}. Contractions are observed less often, so the consumer has fewer opportunities to learn about them.

Although the contraction-state price of risk is higher and more persistent, the unconditional price of risk more closely resembles the expansion-state price. This reflects the unequal weights attached to expansion- and contraction-state values when forming unconditional moments. The conditional moments in (36) are weighted by the ergodic probabilities F_h^U and F_l^U, so expansion-state moments get a weight roughly 20 times that of the contraction-state values. Thus, the expansion-state price of risk is more influential for the unconditional price of risk.

Conditional prices of risk are not closely linked to the unconditional Sharpe ratios reported above, but they are interesting because they have implications for the costs of business cycles. We want to study that connection later.

Next, we explore why the RE price of risk is so much larger. First, we examine whether this reflects distortions to the mean or variance of the discount factor. The ratio of risk prices can be written as the product of a ratio of means and a ratio of standard deviations,

$$\frac{PR_t^{RE}}{PR_t^s} = \frac{\sigma_t^a(m_{t+1}^*)}{\sigma_t^s(m_{t+1})} \frac{E_t^s(m_{t+1})}{E_t^a(m_{t+1}^*)}. \tag{38}$$

Figure 4 illustrates the two terms, the left panel showing $E_t^s(m_{t+1})/E_t^a(m_{t+1}^*)$ and the right $\sigma_t^a(m_{t+1}^*)/\sigma_t^s(m_{t+1})$. The latter is clearly much more important; RE prices of risk are higher principally because the RE discount factor m_t^* is much more variable than the consumer's IMRS. The twisting of the mean makes only a small contribution to a higher price of risk.
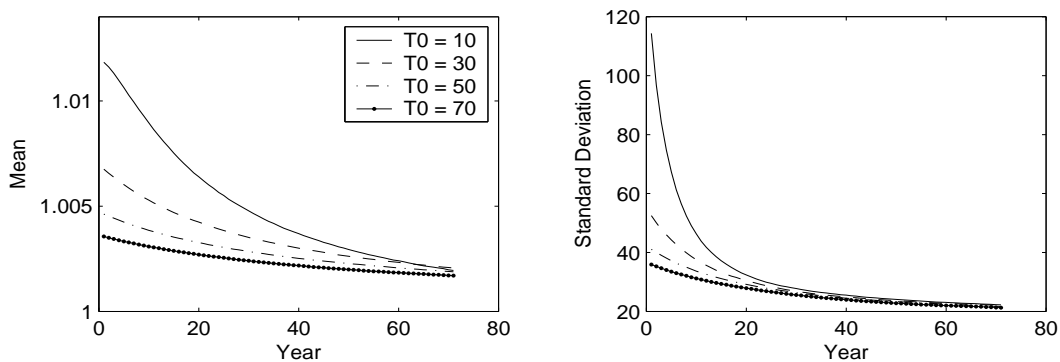

Figure 4: Ratio of Means and Standard Deviations

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To determine why m^*_{t+1} is more volatile than m_{t+1}, we expand the mean-square of m^*_{t+1} as

$$E^a_t(m^{*2}_{t+1}) = E^a_t(\tau^2_{t+1}m^2_{t+1}), \tag{39}$$
$$= E^a_t(\tau^2_{t+1})E^a_t(m^2_{t+1}) + cov^a_t(\tau^2_{t+1}, m^2_{t+1}),$$

where τ_{t+1} denotes the Radon-Nikodým derivative. The two mean-square terms on the right-hand side are

$$E^a_t(m^2_{t+1}) = \sum_{i=1}^{2} F^U_i \left[\sum_{j=1}^{2} F_{ij} m^2_j(t+1) \right], \tag{40}$$

$$E^a_t(\tau^2_{t+1}) = \sum_{i=1}^{2} F^U_i \left[\sum_{j=1}^{2} F_{ij} \left(\frac{F_{ij}(t)}{F_{ij}} \right)^2 \right],$$

and the covariance term can be evaluated as a residual. After normalizing by $E^s_t(m^2_{t+1})$, we can express the relative mean-square of the two discount factors as

$$\frac{E^a_t(m^{*2}_{t+1})}{E^s_t(m^2_{t+1})} = E^a_t(\tau^2_{t+1}) \frac{E^a_t(m^2_{t+1})}{E^s_t(m^2_{t+1})} + \frac{cov^a_t(\tau^2_{t+1}, m^2_{t+1})}{E^s_t(m^2_{t+1})}. \tag{41}$$

Figure 5 depicts each of the terms in this decomposition. Solid lines record the left-hand term, $E^a_t(m^{*2}_{t+1})/E^s_t(m^2_{t+1})$, which is the object we want to decompose. Dashed lines illustrate the mean square of the Radon-Nikodým derivative, $E^a_t(\tau^2_{t+1})$, dashed-dotted lines show the ratio of the mean-square of the consumer's IMRS under the two probability measures, $E^a_t(m^2_{t+1})/E^s_t(m^2_{t+1})$, and solid-dotted lines represent the covariance term, $cov^a_t(\tau^2_{t+1}, m^2_{t+1})/E^s_t(m^2_{t+1})$. The ratio of the mean-sqare of the consumer's true IMRS is always close to 1, and the covariance term is visually hard to distinguish from zero. That means

$$\frac{E^a_t(m^{*2}_{t+1})}{E^s_t(m^2_{t+1})} \doteq E^a_t(\tau^2_{t+1}), \tag{42}$$

so that the magnification of the volatility of m^*_{t+1} relative to m_{t+1} is due almost entirely to variation in the Radon-Nikodým derivative.
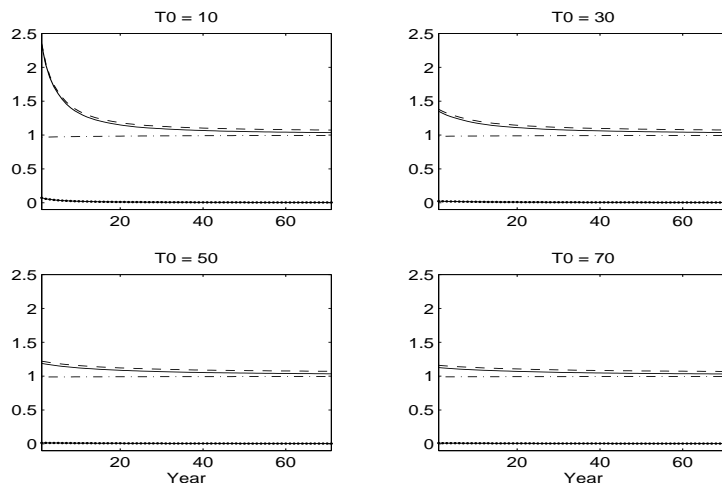
Figure 5: Decomposing the Relative Mean-Square of the Two Discount Factors

Finally, we investigate whether uncertainty about expansion- or contraction-state probabilities contributes more to the unconditional mean-square of τ_{t+1}. Equation (40) says that the unconditional mean-square of τ_{t+1} is a weighted average of expansion and contraction state terms with ergodic probabilities as weights. Conditional on being in an expansion, the mean-square of τ_{t+1} is

$$E_t^a(\tau_{t+1}^2 | s_t = h) = \sum_{j=1}^{2} F_{hj} \left(\frac{F_{hj}(t)}{F_{hj}} \right)^2. \tag{43}$$

The analogous contraction-state term is

$$E_t^a(\tau_{t+1}^2 | s_t = l) = \sum_{j=1}^{2} F_{lj} \left(\frac{F_{lj}(t)}{F_{lj}} \right)^2. \tag{44}$$

These terms are weighted by F_h^U and F_l^U, respectively, when forming the unconditional mean square. By inspecting each term, we can identify the component of model uncertainty that is most important for amplifying the price of risk.

In figure 6, solid lines represent the unconditional mean square, dashed lines portray the contraction-state value, and dashed-dotted and green and red lines depict the expansion-state values. The contraction-state value is almost always higher than that for expansions, and it converges to 1 more slowly. Hence, there is more uncertainty about contraction state probabilities and that uncertainty persists longer. But notice how close the expansion-state value is to the unconditional mean square. Once again, that reflects that F_h^U is roughly 20 times larger than F_l^U. The solid line is a weighted average of the other two, with expansions getting a weight of 0.9566 and contractions a weight of 0.0434. Thus, despite the greater uncertainty about contraction probabilities, it is uncertainty about expansion-state probabilities that contributes most to variation in τ_{t+1}.
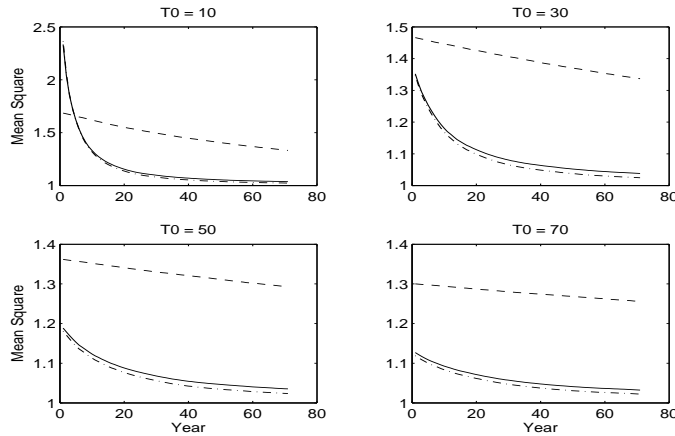

Figure 6: Decomposing the Mean-Square of the Radon-Nikodým Derivative

To summarize, there are two model prices of risk in our economy. Although the subjective price of risk is quite small, the price of risk needed to reconcile returns with rational expectations is quite large. The reason why the RE price of risk is so much larger is that it encompasses both risk aversion and model uncertainty. The modified discount factor m_{t+1}^* involves both the consumers' intertemporal marginal rate of substitution, which encodes risk aversion, and the Radon-Nikodým derivative, which represents uncertainty about the right model for the endowment process. Risk aversion and model uncertainty are both in play, but our consumers are actually very risk tolerant. Their IMRS varies, but only a little. Most of the variation in m_{t+1}^* arises from variation in the Radon-Nikodým derivative, principally from terms connected to uncertainty about expansions. Thus, the high RE price of risk mostly reflects model uncertainty and changing beliefs.

3.2 Asset Returns in the Learning Economy

Thus far we have concentrated on model prices of risk. Now we turn to the properties of asset returns. Figure 7 reports Sharpe ratios for excess returns in each year. Returns are nonstationary under learning, so we initially compute the mean and standard deviation by taking cross-sectional averages of the 5000 trajectories at each date.
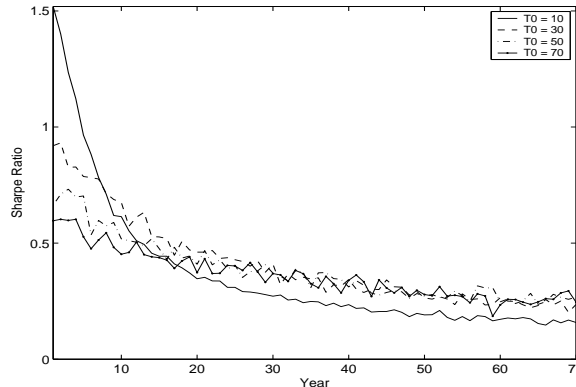

Figure 7: Sharpe Ratios in the Learning Economy

In the first two decades of the simulation, Sharpe ratios are quite a bit higher than those recorded in table 4. They cluster around 0.55 in year 10 of the simulation and 0.4 in year 20. They fall to the neighborhood of 0.25 by year 50, however, and by year 70 they range from 0.16 to 0.27. At least in broad terms, the model reproduces the decline in Sharpe ratios at which table 4 hints.

To make closer contact with the estimates in table 4, we also calculate Sharpe ratios over intervals of time rather than at a point in time. Table 5 reports Sharpe ratios for the full 70-year simulation and also for the first and second halves. Over the full simulation, Sharpe ratios range from 0.3 to 0.4, depending on the prior. They

are higher in the first 35 years of the simulation, clustering around 0.47, and lower in the second half, averaging 0.257. The model Sharpe ratios are therefore a bit too high relative to those found in the data.

Table 5: Model Sharpe Ratios

	$T_0 = 10$	30	50	70
Full Simulation	0.301	0.394	0.378	0.357
First Half	0.443	0.531	0.473	0.436
Second Half	0.190	0.272	0.286	0.281

Figure 8 decomposes the Sharpe ratio into its two components. Solid lines depict mean excess returns in each year, and dashed lines illustrate standard deviations. Here the results are more sensitive to assumptions about the prior, with excess returns being higher on average and more volatile the smaller is T_0. For example, when $T_0 = 10$, the mean excess return starts at around 12 percent and falls eventually to about 3 percent. In contrast, when $T_0 = 70$, the corresponding numbers are 3.5 and 1.5 percent, respectively. Returns are also more volatile when T_0 is small. The standard deviation of excess returns ranges from 10 to 20 percent when $T_0 = 10$, but it is more-or-less constant at 6 percent when $T_0 = 70$. Thus, although Sharpe ratios are reasonably robust across priors, their numerators and denominators are not.
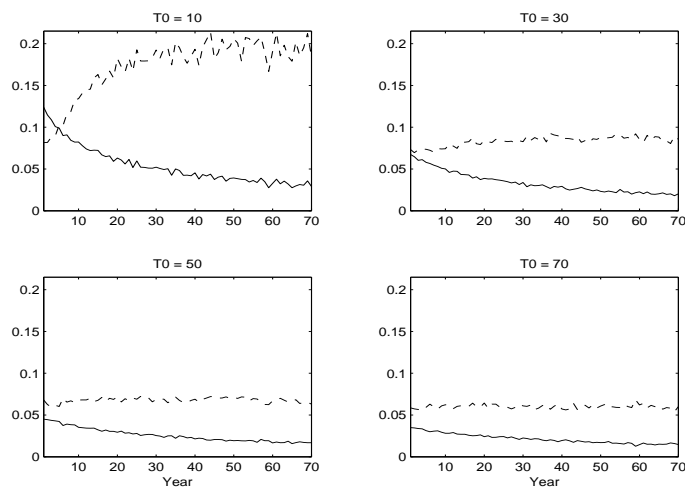

Figure 8: Mean and Standard Deviation of Excess Returns. Solid lines represent the mean and dashed lines the standard deviation.

The value of T_0 influences the consumer's prior in two ways. The consumer is initially more pessimistic when T_0 is small, but his prior is also more diffuse and his outlook is more flexible. Holding constant the detection error probability, the

smaller is T_0, the larger is the set of transition matrices consistent with the available observations, and the more alarming is the worst-case member of that set. The Bayesian detection test becomes more reliable as T_0 increases, and the worst-case model shrinks toward the true model, thus diminishing the degree of initial pessimism. On the other hand, a smaller value of T_0 also makes priors more diffuse, so that beliefs are more responsive to new observations and less firmly anchored to the initial transition matrix. A small T_0 therefore promotes both pessimism and learning.

These two features of the prior – pessimism and diffusion – have countervailing effects on the Sharpe ratio. Greater initial pessimism contributes to higher average excess returns early in the simulation by making the consumer shy away from the risky asset. A more diffuse prior magnifies variation in the Radon-Nikodým derivative, thus contributing to higher volatility of returns. The former increases the Sharpe ratio, while the latter diminishes it. On balance, the two forces roughly cancel, making the Sharpe ratio relatively insensitive to variation in T_0. But assumptions about the prior do influence the numerator and denominator of the Sharpe ratio.

It follows that some versions of our model are more successful than others at matching the equity premium. The equity premium is usually measured not as a mean excess return at a point in time, as in figure 8, but rather as a mean excess return over an interval of time. To make contact with the usual measure, we calculate mean excess returns from the beginning of the simulation through period t. That expanding-sample estimate is shown in figure 9. These estimates are higher than those in figure 8 because high excess returns earned early in the learning transition continue to count later on.
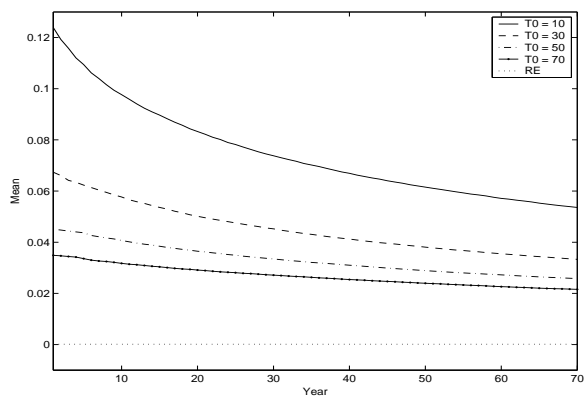

Figure 9: Mean Excess Returns from Year 1 Through Year t

The model succeeds in explaining the equity premium when the representative consumer's prior is sufficiently pessimistic and diffuse. When $T_0 = 10$, the equity premium is around 10 percent in the first decade and then begins to decline, falling to 7.0 percent after 35 years and to 5.3 percent after 70. In addition, the average from years 36 to 70 comes out to 3.7 percent. These numbers accord well with the estimates shown in table 4. Not only does the model match the full-sample average

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of 5.21 percent for 1929-2002, it also well approximates the split-sample averages of 7.08 percent for 1929-1965 and 3.34 percent 1966-2002. Thus, with priors based on a small initial sample, the model can approximate both the average level of the equity premium and its downward trajectory.

The equity premium is a decreasing function of T_0, however, so other versions of the model are less successful. For example, for $T_0 = 50$, the model generates an equity premium of 3.2 percent in the first half of the simulation, 1.95 in the second, and 2.6 percent for the full sample, thus accounting for about half the observed values. Results for $T_0 = 30$ are a bit better than this, those for $T_0 = 70$ a bit worse. Table 6 summarizes these calculations.

Table 6: Model Equity Premia

	$T_0 = 10$	30	50	70
Full Simulation	0.0536	0.0333	0.0258	0.0216
First Half	0.0702	0.0431	0.0321	0.0263
Second Half	0.0370	0.0236	0.0195	0.0169

To match the equity premium, our model must assume that the Depression marked a severe break with the past, shattering the representative consumer's beliefs about the likelihood of expansions and contractions.[23] After a sharp structural break, it is plausible that agents would radically revise beliefs. That they would adopt a diffuse prior is also credible, because they would have had little previous experience in the new environment. It is also conceivable that they would initially guard against model uncertainty by forming robust decision rules. But whether the Depression shattered beliefs to the extent envisioned in our $T_0 = 10$ model is hard to say.

Finally, figure 10 reports results on the risk-free rate. Our main concern here is to confirm that pessimism and learning do not make the risk-free rate excessively volatile, as in some early versions of habit-formation models. The figure demonstrates that pessimism and learning do alter the risk-free rate, but not by very much. The mean return under learning is slightly below its value under rational expectations, but the gap is small and it closes quickly. The standard deviation is higher in the learning economy, but not dramatically higher. Changing beliefs increase the variability of riskless bond prices, but they do not introduce wild swings.
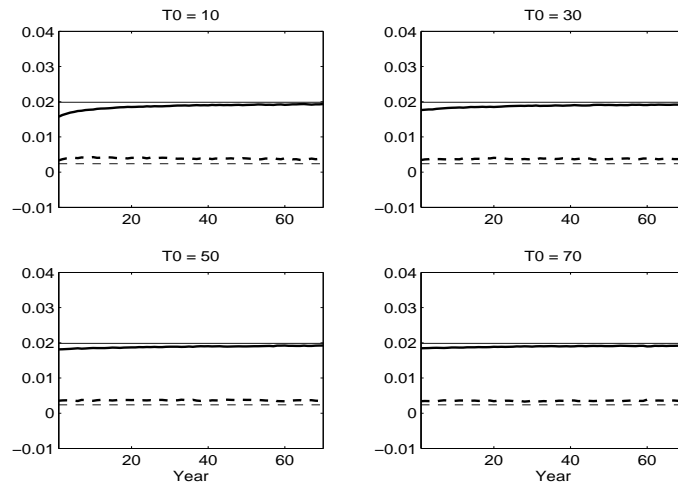

Figure 10: Mean and Standard Deviation of the Risk-Free Rate. Solid lines represent the mean, and dashed lines are standard deviations. Thick lines refer to the learning economy and thin lines to the rational-expectations equilibrium.

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[23]It certainly shattered prevailing opinions among economists. For example, witness the evolution of Keynes's thinking as he passed from the orthodoxy of *A Tract on Monetary Reform* (1923) to *The General Theory of Employment, Interest, and Money* (1936).

4   Concluding Remarks

This paper explores an asset pricing model in which Bayes' law gradually tempers a representative consumer's initial pessimism about conditional distributions of consumption growth. We study an endowment economy in which a representative consumer learns about the transition probabilities of a Markov chain. We endow the consumer with pessimistic prior beliefs that we induce via a robustness calculation and then allow the consumer to update his beliefs via Bayes's theorem. Decision makers' evolving beliefs contribute a component to what a rational expectation econometrician would measure as the stochastic discount factor. We study how the market price of risk, Sharpe ratios, and the equity premium behave as the consumer learns.

The model can explain – at least in broad terms – two asset pricing anomalies. One concerns a conflict between measures of risk aversion inferred from security market data and those derived from surveys or experiments in which individuals are presented with well-understood and well-controlled gambles. The former are usually very high, yet the latter are often quite low. Hansen and Jagannathan (1991) characterize this dissonance in terms of the market price of risk, which they define as the ratio of the standard deviation of a stochastic discount factor to its mean. A high market price of risk emerges from security market data, yet the degree of risk aversion found in surveys suggests a much lower price of risk in models.

In our model, the conflict is explained by noting that there are two market prices of risk that correspond to the two probability measures in play. One uses the representative consumer's subjective transition probabilities to evaluate the mean and standard deviation, and the other uses the true transition probabilities. If one interviewed the representative consumer in our model, his answers would reflect subjective probabilities and would convey a small price of risk. But if one calculated the market price of risk needed to reconcile equilibrium returns with the true transition probabilities, as in a rational expectations model, the estimate would be substantially higher, reflecting the near-arbitrage[24] opportunities seemingly available ex post. A mild degree of risk aversion for investors thereby coexists with a high price of risk estimated from asset return data.

A closely-related conflict involves the fact that arbitrage or near-arbitrage opportunities can often be detected ex post in analyses of security market data, yet investors seem unable to grow rich off them in real time. In the model, this occurs because Euler equations hold ex ante with respect to agents' subjective transition probabilities, but not ex post with respect to realized frequencies. Absence of arbitrage for the agents coexists with ex post violations of Euler equations.

One version of our model, in which priors are based on a small initial sample of data, can also explain the equity premium. Equities are initially undervalued relative

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[24]By 'near arbitrage,' we mean an investment opportunity with a high Sharpe ratio. Cochrane and Saá-Requejo (2000) call this a 'good deal.' Under rational expectations, agents would have to be very risk averse to shy away from good deals.

to the true transition probabilities because the representative consumer over-estimates the likelihood of another Depression. The representative consumer becomes less pessimistic as he learns, and equity prices rise, correcting the initial undervaluation. A high ex-post equity premium emerges because realized excess returns systematically exceed expected excess returns, reflecting the prevalence of unexpected capital gains along the transition path. The learning process takes a long time because particular transitions occur rarely and therefore take substantial time to learn about. The model equity premium falls as agents learn, in accordance with evidence reported by Blanchard (1993), Jagannathan, McGratten, and Scherbina (2000), Fama and French (2002), DeSantis (2004), and others that the actual equity premium is declining. These authors also emphasize that expected excess stock returns are now lower than historical averages, which is also an implication of our model.[25]

We accomplish all of this within a model in which the discount factor β is slightly less than 1 and the coefficient of relative risk aversion α is slightly above 0. Our investors are not too impatient and only mildly risk averse. Most of the action is due to pessimism and learning. Model uncertainty is more important than risk.

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[25]Compare the end-of-simulation values in figures 8 and 9.

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